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A nontrivial solvable noncommutative ϕ^3 model in 4 dimensions

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Abstract: We study the quantization of the noncommutative selfdual ϕ^3 model in 4 dimensions, by mapping it to a Kontsevich model. The model is shown to be renormalizable, provided one additional counterterm is included compared to the 2-dimensional case [1] which can be interpreted as divergent shift of the field ϕ. The known results for the Kontsevich model allow to obtain the genus expansion of the free energy and of any n-point function, which is finite for each genus after renormalization. No coupling constant or wave-function renormalization is required. A critical coupling is determined, beyond which the model is unstable. This provides a nontrivial interacting NC field theory in 4 dimensions.

Keywords: Integrable Field Theories, Matrix Models, Non-Commutative Geometry, Renormalization Regularization and Renormalons.

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Contents

1. Introduction

This paper is an extension of our previous work [1] on the noncommutative (NC) Euclidean selfdual ϕ^3 model. In [1] we considered the 2-dimensional case, and showed that this model can be renormalized and essentially solved using matrix model techniques. This was achieved by mapping it to the Kontsevich model.

The map from the selfdual NC ϕ^3 model to the Kontsevich model exists for any even dimension, however the eigenvalues and their degeneracy of the corresponding Kontsevich model depend on the dimension of the underlying space. Therefore the properties of the model and in particular its renormalizability must be studied separately in different dimensions. In the present paper we elaborate the 4-dimensional case. Generalizing [1], we prove renormalizability and obtain closed expressions for the genus expansion of the free energy F, based on results of [2, 3] for the Kontsevich model. The general n-point functions can then be computed by taking derivatives of F, in an explicit way for diagonal entries ϕ_{ii} and somewhat implicitly for the general case. As an example, we work out the genus 0 contributions for the 1− and 2-point functions.

It turns out that as in the 2-dimensional case, the required renormalization is determined by the genus 0 sector only, and can be computed explicitly. We show that all

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contributions in a genus expansion of any n-point function are finite and well-defined after renormalization, for finite coupling. A linear (tadpole) counterterm must be introduced which is linearly divergent, while mass and a further counterterm are logarithmically divergent. The coupling constant does not run as expected. This implies but is stronger than perturbative renormalization. We thus obtain a fully renormalized and essentially solvable NC field theory with nontrivial interaction, by applying the rich structure of the Kontsevich model related to integrable models (KdV flows) and Virasoro constraints [3, 4, 2]. The model is free of UV/IR diseases due to the confining oscillator potential introduced in [5 − 8].

As in [1], these results are obtained starting with purely imaginary coupling constants $i\lambda$, but allow analytic continuation to real coupling as long as $|\lambda|$ is small enough. If λ is larger than some critical value, the model becomes unstable. We determine the corresponding critical coupling, which is interpreted as instability induced by the finite potential barrier.

This paper is largely parallel to our previous work [1] on the 2-dimensional case. Therefore we will be short in certain issues which have already been discussed there, and which apply without change. Nevertheless, the present paper is essentially self-contained. In section 2 we define the ϕ^3 model under consideration, and rewrite it as Kontsevich model. There is one additional counterterm compared to [1]. We then briefly recall the most important facts about the Kontsevich model in section 3. Renormalization and finiteness are established in section 4.1, which is the main result of this paper. The 2-point function at genus 0 is worked out explicitly in section 4.2, and the computation of the general n-point function is discussed in section 4.3. The critical point is determined in section 4.5. We conclude with a discussion and outlook.

2. The noncommutative ϕ^3 model

We consider the 4-dimensional scalar noncommutative ϕ^3 model, with an additional oscillator-type potential in order to avoid the problem of UV/IR mixing. The model is defined by the action

$$\tilde{S} = \int_{\mathbb{R}^4_\theta} \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{\mu^2}{2}\phi^2 + \Omega^2(\tilde{x}_i\phi)(\tilde{x}_i\phi) + \frac{i\tilde{\lambda}}{3!}\,\phi^3 \tag{2.1}$$

on the 4-dimensional quantum plane, which is generated by self-adjoint operators[1] x_i satisfying the canonical commutation relations

$$[x_i, x_j] = i\theta_{ij}, \tag{2.2}$$

$i, j = 1, 2, 3, 4$. We also introduce

$$\tilde{x}_i = \theta^{-1}_{ij}x_j, \qquad [\tilde{x}_i, \tilde{x}_j] = i\theta^{-1}_{ji} \tag{2.3}$$

[1]We ignore possible operator-technical subtleties here, since the model will be regularized using a cutoff N below.

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assuming that θ_{ij} is nondegenerate. The dynamical object is the scalar field $\phi = \phi^\dagger$, which is a self-adjoint operator acting on the representation space $\mathcal{H}$ of the algebra (2.2). The term $\Omega^2(\tilde{x}_i\phi)(\tilde{x}_i\phi)$ is included following [5–8], making the model covariant under Langmann-Szabo duality, and taking care of the UV/IR mixing. We choose to write the action with an imaginary coupling $i\tilde{\lambda}$, assuming $\tilde{\lambda}$ to be real. The reason is that for real coupling $\tilde{\lambda}' = i\tilde{\lambda}$, the potential would be unbounded from above and below, and the quantization would seem ill-defined. We will see however that the quantization is completely well-defined for imaginary $i\tilde{\lambda}$, and allows analytic continuation to real $\tilde{\lambda}' = i\tilde{\lambda}$ in a certain sense which will be made precise below. Therefore we accept for now that the action $\tilde{S}$ is not necessarily real.

Using the commutation relations (2.2), the derivatives ∂_i can be written as inner derivatives $\partial_i f = -i[\tilde{x}_i, f]$. Therefore the action can be written as

$$\tilde{S} = \int -(\tilde{x}_i\phi\tilde{x}_i\phi - \tilde{x}_i\tilde{x}_i\phi\phi) + \Omega^2\tilde{x}_i\phi\tilde{x}_i\phi + \frac{\mu^2}{2}\phi^2 + \frac{i\tilde{\lambda}}{3!}\,\phi^3 \tag{2.4}$$

using the cyclic property of the integral. For the “self-dual” point $\Omega = 1$, this action simplifies further to

$$\tilde{S} = \int (\tilde{x}_i\tilde{x}_i + \frac{\mu^2}{2})\phi^2 + \frac{i\tilde{\lambda}}{3!}\,\phi^3 \;=\; Tr\Big(\frac{1}{2}J\phi^2 + \frac{i\lambda}{3!}\,\phi^3\Big). \tag{2.5}$$

Here we replaced the integral by $\int = (2\pi\theta)^2 Tr$, and introduce

$$J = 2(2\pi\theta)^2(\sum_i \tilde{x}_i\tilde{x}_i + \frac{\mu^2}{2}), \qquad \lambda = (2\pi\theta)^2\tilde{\lambda}. \tag{2.6}$$

We assume that θ_{ij} has the canonical form $\theta_{12} = -\theta_{21} =: \theta \;= \theta_{34} = -\theta_{43}$. Then J is essentially the Hamiltonian of a 2-dimensional quantum mechanical harmonic oscillator, which in the usual basis of eigenstates diagonalizes as

$$J|n_1, n_2\rangle = 8\pi^2\theta\,(n_1 \;+ n_2 \;+ 1 + \frac{\mu^2\theta}{2})|n_1, n_2\rangle, \qquad n_i \in \{0, 1, 2, \ldots\}. \tag{2.7}$$

To simplify the notation, we will use the convention

$$n \equiv (n_1, n_2), \qquad \underline{n} \equiv n_1 + n_2 \tag{2.8}$$

throughout this paper, keeping in mind that n denotes a double-index.

In order to quantize the theory, we need to include a linear counterterm $-Tr(i\lambda)a\,\phi$ to the action (the explicit factor $i\lambda$ is inserted to keep most quantities real), and — as opposed to the 2-dimensional case [1] — we must also allow for a divergent shift

$$\phi \to \phi + i\lambda c \tag{2.9}$$

of the field ϕ. These counterterms are necessary to ensure that the local minimum of the cubic potential remains at the origin after quantization. The latter shift implies in particular that the linear counterterm picks up a contribution $-Tr(i\lambda)(a + cJ)\phi$ from the quadratic term. Therefore the linear term should be replaced by $-Tr(i\lambda)A\phi$ where

$$A = a + cJ, \tag{2.10}$$

while the other effects of this shift $\phi \to \phi + i\lambda c$ can be absorbed by a redefinition of the coupling constants (which we do not keep track of). We are thus led to consider the action

$$S = Tr\Big(\frac{1}{2}J\phi^2 + \frac{i\lambda}{3!}\,\phi^3 - (i\lambda)A\phi - \frac{1}{3(i\lambda)^2}J^3 - JA\Big). \tag{2.11}$$

involving the constants $i\lambda$, a, c and μ^2. The additional constant terms in (2.11) are introduced for later convenience. By suitable shifts in the field ϕ, one can now either eliminate the linear term or the quadratic term in the action,

$$S = Tr\Big(-\frac{1}{2i\lambda}M^2\tilde{\phi} + \frac{i\lambda}{3!}\,\tilde{\phi}^3\Big) = Tr\Big(\frac{1}{2}MX^2 + \frac{i\lambda}{3!}\,X^3 - \frac{1}{3(i\lambda)^2}M^3\Big) \tag{2.12}$$

where[2]

$$\tilde{\phi} = \phi + \frac{1}{i\lambda}J \;=\; X + \frac{1}{i\lambda}M \tag{2.13}$$

and

$$M \;=\; \sqrt{J^2 + 2(i\lambda)^2 A} = \sqrt{\tilde{J}^2 + 2(i\lambda)^2 a - (i\lambda)^4 c^2} \tag{2.14}$$

$$\tilde{J} \;=\; J + (i\lambda)^2 c. \tag{2.15}$$

This has precisely the form of the Kontsevich model [3]. Coupling the field linearly to the source will be very useful for computing correlation functions. In particular, this can be compared directly with the 2-dimensional case considered in [1], however $\tilde{J}_k$ plays now the role of J_k, and a further constant c has been introduced.

2.1 Quantization and equations of motion

The quantization of the model (2.11) resp. (2.12) is defined by an integral over all Hermitian $N^2 \times N^2$ matrices ϕ, where N serves as a UV cutoff. The partition function is defined as

$$Z(M) = \int D\tilde{\phi}\, \exp(-Tr\Big(-\frac{1}{2i\lambda}M^2\tilde{\phi} + \frac{i\lambda}{3!}\,\tilde{\phi}^3\Big)) = e^{F(M)}, \tag{2.16}$$

which is a function of the eigenvalues of M resp. $\tilde{J}$. Since N is finite, we can freely switch between the various parametrizations (2.11), (2.12) involving M, J, ϕ, or $\tilde{\phi}$. Correlators or "n-point functions" are defined through

$$\langle \phi_{i_1 j_1} \dots \phi_{i_n j_n} \rangle = \frac{1}{Z} \int D\phi\, \exp(-S)\, \phi_{i_1 j_1} \dots . \phi_{i_n j_n}, \tag{2.17}$$

keeping in mind that each i_n denotes a double-index (2.8). The (degenerate) spectrum of J resp. M is given by (2.7) resp. (2.14). The nontrivial task is to show that all n-point functions have a well-defined and hopefully nontrivial limit $N \to \infty$, so that the "low-energy physics" is well-defined and independent of the cutoff.

[2] for the quantization, the integral for the diagonal elements is then defined via analytical continuation, and the off-diagonal elements remain hermitian since J is diagonal.

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Using the symmetry $Z(M) = Z(U^{-1}MU)$ for $U \in \mathrm{U}(N^2)$, we can assume that M is diagonalized with (ordered) eigenvalues m_i. In fact, Z depends only on the eigenvalues of M^2 resp. $\tilde{J}^2$. There is a residual $\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(3)\times\ldots$ symmetry, reflecting the degeneracy of J. This implies certain obvious "index conservation laws", such as $\langle\phi_{kl}\rangle = \delta_{kl}\langle\phi_{ll}\rangle$ etc.

In order to have a well-defined limit $N \to \infty$, we must require in particular that the 2-point function $\langle\phi_{ij}\phi_{kl}\rangle$ and also the one-point function $\langle\phi_{kl}\rangle$ have a well-defined limit. We therefore impose the renormalization conditions

$$\langle\phi_{00}\phi_{00}\rangle = \frac{1}{2\pi}\frac{1}{\mu_R^2\theta + 1}, \tag{2.18}$$

$$\langle\phi_{00}\rangle = 0 \tag{2.19}$$

together with the requirement that the index dependence at least of the 2-point function is nontrivial. This will uniquely determine the renormalization of a, c and μ^2, while λ receives only finite quantum corrections which will not be computed here.

Quantum equations of motion and correlators. Noting that the field $\tilde{\phi}$ couples linearly to M^2 resp. $\tilde{J}^2$ in (2.16), one can compute insertions of a diagonal factor $\tilde{\phi}_{kk}$ in a correlator by acting with the derivative operator $2i\lambda\frac{\partial}{\partial\tilde{J}_k^2}$ on Z resp. F. More general non-diagonal insertions $\tilde{\phi}_{kl}$ can also be obtained in principle, as discussed in section 4.3. However, using various standard manipulations of the path integral (2.16) one can derive directly a number of nontrivial identities for the n-point functions. Since their derivation as given in [1] is independent of the eigenvalues of $\tilde{J}$, we simply write down the most important identities here, noting that $\tilde{J}$ must be used instead of J. In particular, one finds for the propagator

$$\langle\tilde{\phi}_{kl}\tilde{\phi}_{lk}\rangle = \frac{2i\lambda}{m_k^2 - m_l^2}\langle\tilde{\phi}_{kk} - \tilde{\phi}_{ll}\rangle \tag{2.20}$$

for $k \neq l$ (no sum), where m_k denotes the eigenvalues of M. Recalling that $\tilde{\phi} = \phi + \frac{1}{i\lambda}J$, this gives

$$\langle\phi_{kl}\phi_{lk}\rangle = \frac{2}{\tilde{J}_k + \tilde{J}_l} + \frac{2i\lambda}{\tilde{J}_k^2 - \tilde{J}_l^2}\langle\phi_{kk} - \phi_{ll}\rangle \tag{2.21}$$

noting that $\tilde{J}_k^2 - \tilde{J}_l^2 = m_k^2 - m_l^2$ using (2.14). The first term is the free contribution, and the second the quantum correction. This "only" requires the 1-point functions

$$\langle\tilde{\phi}_{kk}\rangle = 2i\lambda\frac{\partial}{\partial m_k^2}\ln\tilde{Z}(m) = \frac{1}{i\lambda}J_k + \langle\phi_{kk}\rangle \tag{2.22}$$

which can be obtained from the Kontsevich model, as we will show in detail. Furthermore, one can derive

$$\frac{m_k^2}{\lambda^2} = -\langle\tilde{\phi}_{kk}^2\rangle - (2i\lambda)\sum_{l, l\neq k}\frac{\langle\tilde{\phi}_{kk} - \tilde{\phi}_{ll}\rangle}{m_k^2 - m_l^2}. \tag{2.23}$$

These manipulations can be generalized [1]. In particular, certain 3-point functions can be expressed exactly in terms of 1- and 2-point functions, etc. We will not repeat these considerations here. Instead, the finiteness of general correlation functions will be established directly, by showing that the appropriate derivatives of the (connected) generating function $F(\tilde{J}) = \ln Z(\tilde{J})$ are finite and well-defined after renormalization. This could also be used to demonstrate that the model is not free, in contrast to [9]. We will not bother to elaborate this, since the model will be renormalized for finite coupling, and the lowest nontrivial term in an expansion in λ is manifestly finite.

3. Some useful facts for the Kontsevich model

The Kontsevich model is defined by

$$Z^{Kont}(\tilde{M}) = e^{F^{Kont}} = \frac{\int dX \exp\left\{Tr\left(-\frac{\tilde{M}X^2}{2} + i\frac{X^3}{6}\right)\right\}}{\int dX \exp\left\{-Tr\left(\frac{\tilde{M}X^2}{2}\right)\right\}} \tag{3.1}$$

where $\tilde{M}$ is a given hermitian $N^2 \times N^2$ matrix, and the integral is over Hermitian $N^2 \times N^2$ matrices X. This model has been introduced by Kontsevich [3] as a combinatorial way of computing certain topological quantities (intersection numbers) on moduli spaces of Riemann surfaces with punctures, which in turn were related to the partition function of the general one-matrix model by Witten [10]. It turns out to have an extremely rich structure related to integrable models (KdV flows) and Virasoro constraints, and was studied using a variety of techniques. For our purpose, the most important results are those of [3, 4, 2] which provide explicit expressions for the genus expansion of the free energy. Note that λ can be introduced via

$$Z^{Kont}(\tilde{M}) = \frac{\int dX \exp\left\{Tr(-\frac{\tilde{M}X^2}{2} + i\frac{X^3}{6})\right\}}{\int dX \exp\left\{-Tr(\frac{\tilde{M}X^2}{2})\right\}} = \frac{\int d\tilde{X} \exp\left\{Tr(-\frac{M\tilde{X}^2}{2} - i\lambda\frac{\tilde{X}^3}{6})\right\}}{\int d\tilde{X} \exp\left\{-Tr(\frac{M\tilde{X}^2}{2})\right\}}, \tag{3.2}$$

where $X = -\lambda^{1/3}\tilde{X}$, $M = \lambda^{2/3}\tilde{M}$, which allows to obtain the analytic continuation in λ.

The matrix integral in (3.1) and its large N limit can be defined rigorously in terms of its asymptotic series. This involves in a crucial way the variables [3]

$$t_r := -(2r+1)!!\ \theta_{2r+1}, \qquad \theta_r := \frac{1}{r} Tr\tilde{M}^{-r}. \tag{3.3}$$

One can then rigorously define [3, 2] the large N limit of the partition function $Z^{Kont}(\tilde{M})$, which turns out to be a function of these new variables only, $Z^{Kont}(\tilde{M}) = Z^{Kont}(\theta_i)$. More precisely, each order in perturbation theory is a polynomial in finitely many of the variables t_r, and becomes independent of N for N large enough. This provides a rigorous definition of the ϕ^3 model. For more details we refer to [3, 2, 1].

Without renormalization (i.e. for finite or zero a), θ_r is linearly divergent for $r = 1$, logarithmically divergent for $r = 2$, and finite for $r \geq 3$. This is a first indication that the model requires renormalization.

A further crucial fact is the existence of a

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Genus expansion. As usual for matrix models, one can consider the genus expansion

$$\ln Z^{Kont} = F^{Kont} = \sum_{g\geq 0} F_g^{Kont} \tag{3.4}$$

by drawing the Feynman diagrams on a suitable Riemann surface. In principle, this genus expansion can be obtained as a $\frac{1}{N}$ expansion by introducing an explicit factor N in the action, so that the action takes the form

$$S' = TrN(-\frac{1}{2}M'^2\phi' + \frac{1}{3!}\,\phi'^3). \tag{3.5}$$

However, it was shown in [2] that the F_g^{Kont} can also be computed using the KdV equations and the Virasoro constraints, which allows to find closed expressions for given g. It is useful to use the following set of variables:

$$I_k(u_0, t_i) = \sum_{p\geq 0} t_{k+p}\frac{u_0^p}{p!} \tag{3.6}$$

where u_0 is given by the solution of the implicit equation

$$u_0 = I_0(u_0, t_i). \tag{3.7}$$

We note that I_k can be resummed as

$$I_k(u_0, t_i) = -(2k-1)!!\sum_i \frac{1}{(\tilde{m}_i^2 - 2u_0)^{k+\frac{1}{2}}}, \tag{3.8}$$

in particular

$$u_0 = -\sum_i \frac{1}{\sqrt{\tilde{m}_i^2 - 2u_0}} = I_0. \tag{3.9}$$

These variables turn out to be more useful for our purpose than the t_r, since the quantities $\tilde{m}_i^2 - 2u_0$ will be finite in the renormalized model, while the t_r are not. Using the KdV equations, [2] find the following explicit formulas:

$$F_0^{Kont} = \frac{u_0^3}{6} - \sum_k \frac{u_0^{k+2}}{k+2}\frac{t_k}{k!} + \frac{1}{2}\sum_k \frac{u_0^{k+1}}{k+1}\sum_{a+b=k}\frac{t_a}{a!}\frac{t_b}{b!} \tag{3.10}$$

$$F_1^{Kont} = \frac{1}{24}\ln\frac{1}{1-I_1}, \tag{3.11}$$

$$F_2^{Kont} = \frac{1}{5760}\left[5\frac{I_4}{(1-I_1)^3} + 29\frac{I_3 I_2}{(1-I_1)^4} + 28\frac{I_2^3}{(1-I_1)^5}\right], \tag{3.12}$$

etc. All F_g^{Kont} with $g \geq 2$ are given by *finite* sums of polynomials in $I_k/(1-I_1)^{\frac{2k+1}{3}}$, the number of which is $p(3g-3)$ with $p(n)$ being the number of partitions of n.

An alternative form of F_0^{Kont} can be obtained by solving directly the "master-equation" (2.23) at genus 0. This leads to [4]

$$\begin{aligned} F_0^{Kont} = {} & \frac{1}{3}\sum_i \tilde{m}_i^3 - \frac{1}{3}\sum_i(\tilde{m}_i^2 - 2u_0)^{3/2} - u_0\sum_i(\tilde{m}_i^2 - 2u_0)^{1/2} \\ & + \frac{u_0^3}{6} - \frac{1}{2}\sum_{i,k}\ln\left\{\frac{(\tilde{m}_i^2 - 2u_0)^{1/2} + (\tilde{m}_k^2 - 2u_0)^{1/2}}{\tilde{m}_i + \tilde{m}_k}\right\} \end{aligned} \tag{3.13}$$

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which is equivalent to (3.10) but more useful in our context. All sums over double-indices (2.8) here and in the following are to be interpreted as

$$\sum_i \equiv \sum_{i_1,i_2=0}^{N-1}$$

truncating the harmonic oscillators in (2.7). The parameter u_0 is given by the implicit equation (3.9). This constraint can alternatively be obtained by considering $F^{Kont}(\tilde{m}_i; u_0)$ with u_0 as an independent variable, since its equation of motion

$$\frac{\partial}{\partial u_0} F^{Kont}(\tilde{m}_i; u_0) = \frac{1}{2}(u_0 - I_0)^2 = 0 \tag{3.14}$$

reproduces the constraint. After renormalization, all sums will be convergent for the physical observables as $N \to \infty$.

4. Applying Kontsevich to the ϕ^3 model

We need

$$Z = Z^{Kont}[\tilde{M}] Z^{free}[\tilde{M}] \exp(\frac{1}{3(i\lambda)^2} TrM^3) \tag{4.1}$$

where

$$Z^{free}[\tilde{M}] = e^{F_{free}} = \int dX \exp\left(-Tr\left(\frac{\tilde{M}X^2}{2}\right)\right) = \prod_i \frac{1}{\sqrt{\tilde{m}_i}} \prod_{i<j} \frac{2}{\tilde{m}_i + \tilde{m}_j} \tag{4.2}$$

up to irrelevant constants, so that

$$F_{free} = -\frac{1}{2} \sum_{i,j} \ln(\tilde{m}_i + \tilde{m}_j) \quad (+const). \tag{4.3}$$

Therefore

$$\begin{aligned} F_0 &:= F_0^{Kont} + F_{free} + \frac{1}{3(i\lambda)^2} TrM^3 \\ &= -\frac{1}{3} \sum_i \sqrt{\tilde{m}_i^2 - 2u_0}^3 - u_0 \sum_i \sqrt{\tilde{m}_i^2 - 2u_0} \\ &\quad + \frac{u_0^3}{6} - \frac{1}{2} \sum_{i,k} \ln(\sqrt{\tilde{m}_i^2 - 2u_0} + \sqrt{\tilde{m}_k^2 - 2u_0}). \end{aligned} \tag{4.4}$$

In the present case, the eigenvalues $\tilde{m}_i$ are given by (3.2), (2.14)

$$\tilde{m}_i = \lambda^{-2/3} \sqrt{J_i^2 + 2(i\lambda)^2 A_i}, \tag{4.5}$$

and the model will be ill-defined without renormalization since u_0 is linearly divergent. However, note that only the combinations $\sqrt{\tilde{m}_i^2 - 2u_0}$ enter in (3.8) and (4.4), which can be rewritten using (2.14) as

$$\sqrt{\tilde{m}_i^2 - 2u_0} = \lambda^{-2/3} \sqrt{\tilde{J}_k^2 + 2b} \tag{4.6}$$

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where

$$b = (i\lambda)^2 a - \lambda^{4/3}\, u_0 - (i\lambda)^4 c^2/2. \tag{4.7}$$

The point is that both $\tilde{J}$ and b will be finite after renormalization, rendering the model well-defined.

The model and therefore the free energy F depends a priori on the parameters a, c, λ, and J_k resp. $\tilde{J}_k$ which in turn contains μ^2. In the form (4.4), new parameters b and u_0 have appeared, which are implicitly determined by (4.7) and the constraint (3.9),

$$(b - (i\lambda)^2 a + (i\lambda)^4 c^2/2) = -\sum_i \frac{(i\lambda)^2}{\sqrt{\tilde{J}_i^2 + 2b}}\,. \tag{4.8}$$

Eliminating u_0 by (4.7), the genus 0 contribution to the partition function (4.4) takes the form

$$\begin{aligned} F_0 = \ln Z_{g=0} = \frac{(i\lambda)^{-2}}{3}\sum_i \sqrt{\tilde{J}_i^2 + 2b}^{\,3} + \Big(a - \frac{b}{(i\lambda)^2} - (i\lambda)^2 c^2/2\Big)\sum_i \sqrt{\tilde{J}_i^2 + 2b} \\ + \frac{(i\lambda)^2}{6}\Big(a - \frac{b}{(i\lambda)^2} - (i\lambda)^2 c^2/2\Big)^3 - \frac{1}{2}\sum_{i,k}\ln\Big(\lambda^{-\frac{2}{3}}\sqrt{\tilde{J}_i^2 + 2b} + \lambda^{-\frac{2}{3}}\sqrt{\tilde{J}_k^2 + 2b}\Big)\,. \end{aligned} \tag{4.9}$$

We consider $F = F(\tilde{J}^2)$ as a function of (the eigenvalues of) $\tilde{J}^2$ from now on, with further parameters a, c, λ, while b is implicitly determined by (4.8). Since $\tilde{J}_k$ only enters through the combination $\sqrt{\tilde{J}_i^2 + 2b}$, we note that the eigenvalues can be analytically continued as long as this square-root is well-defined.

We can now compute various n-point functions by taking partial derivatives of $F = \sum_g F_g$ (where $F_g = F_g^{Kont}$ for $g \geq 1$) w.r.t. $\tilde{J}_k^2$, as indicated in section 2.1. For the "diagonal" n-point functions $\langle\tilde{\phi}_{ii}\ldots\tilde{\phi}_{kk}\rangle$, this amounts to varying the eigenvalues of $\tilde{J}_k^2$. In doing so, we must be careful to keep the parameters a, c constant since they determine the model, and note that b depends implicitly on $\tilde{J}_k^2$ through the constraint (4.8).

Some of these computations simplify by the following observation: The constraint (4.8) for b arises automatically through the e.o.m as in (3.14), noting that

$$\frac{\partial}{\partial\tilde{J}_i^2}F_0(\tilde{J}_i^2) = \frac{\partial}{\partial\tilde{J}_i^2}F_0(\tilde{J}_i^2; b) + \frac{\partial}{\partial b}F_0(\tilde{J}_i^2; b)\frac{\partial}{\partial\tilde{J}_i^2}b = \frac{\partial}{\partial\tilde{J}_i^2}F_0(\tilde{J}_i^2; b) \tag{4.10}$$

using

$$\frac{\partial}{\partial b}\,F_0(\tilde{J}_i^2; b) = -\frac{1}{2}\Big(\big(\frac{b}{(i\lambda)^2} - a + (i\lambda)^2 c^2/2\big) + \sum_i \frac{1}{\sqrt{\tilde{J}_i^2 + 2b}}\Big)^2 = 0. \tag{4.11}$$

Thus for derivatives of order ≤ 2 w.r.t. $\tilde{J}_k^2$, we can simply ignore b and treat it as independent variable, since the omitted terms (4.11) vanish anyway once the constraint is imposed.

4.1 Renormalization and finiteness

The 1-point function We can now determine the required renormalization of a and c, by considering the one-point function. Using (4.9), (4.11) and (4.8), the genus zero contribution is

$$\begin{aligned}\langle\tilde{\phi}_{kk}\rangle_{g=0} &= 2i\lambda\frac{\partial}{\partial\tilde{J}_k^2}F_0(\tilde{J}^2)\\ &= \frac{1}{i\lambda}\sqrt{\tilde{J}_k^2+2b}+\frac{i\lambda}{\sqrt{\tilde{J}_k^2+2b}}\Big((a-\frac{b}{(i\lambda)^2}-(i\lambda)^2c^2/2)\\ &\qquad -\sum_{j=0}^{N}\frac{1}{\sqrt{\tilde{J}_k^2+2b}+\sqrt{\tilde{J}_j^2+2b}}\Big)+2i\lambda\frac{\partial b}{\partial\tilde{J}_k^2}\frac{\partial F_0}{\partial b}\\ &= \frac{1}{i\lambda}\sqrt{\tilde{J}_k^2+2b}+\sum_j\frac{(i\lambda)}{\sqrt{\tilde{J}_k^2+2b}\sqrt{\tilde{J}_j^2+2b}+(\tilde{J}_j^2+2b)}\end{aligned}\tag{4.12}$$

and therefore

$$\begin{aligned}\langle\phi_{kk}\rangle_{g=0} &= \langle\tilde{\phi}_{kk}\rangle_{g=0}-\frac{J_k}{i\lambda}\\ &= \frac{1}{i\lambda}(\sqrt{\tilde{J}_k^2+2b}-\tilde{J}_k)+(i\lambda)c+\sum_j\frac{(i\lambda)}{\sqrt{\tilde{J}_k^2+2b}\sqrt{\tilde{J}_j^2+2b}+(\tilde{J}_j^2+2b)},\end{aligned}\tag{4.13}$$

which must be finite and well-defined in the limit $N\to\infty$.

As opposed to the 2-dimensional case, we now have to face a logarithmic divergence in the sum on the rhs. We note that $\tilde{J}_k$ resp. J_k depends only on the combination

$$\underline{k}:=k_1+k_2\tag{4.14}$$

(recall that $k,j,\ldots$ etc. are 2-component indices (2.8)). In analogy to the usual strategy in renormalization, we consider the Taylor-expansion of

$$\begin{aligned}f(\underline{k}) &:= \sum_j\frac{1}{\sqrt{\tilde{J}_k^2+2b}\sqrt{\tilde{J}_j^2+2b}+(\tilde{J}_j^2+2b)}=f(0)+(\underline{k})\,f'(0)+\cdots\\ &=: f(0)+f_R(\underline{k})\end{aligned}\tag{4.15}$$

in $\underline{k}$, where (keeping only the divergent part)

$$\begin{aligned}f(0) &\approx \sum_j\frac{1}{\tilde{J}_j^2+2b}+const\approx\frac{1}{(8\pi^2\theta)^2}\int_0^N dx_1dx_2\,\frac{1}{(x_1+x_2+const)^2}+const\\ &\approx \frac{1}{(8\pi^2\theta)^2}\ln(N)\end{aligned}\tag{4.16}$$

up to finite corrections. We anticipate here that $\tilde{J}_k$ (4.20) and b (4.21) will be finite after renormalization. Only $f(0)$ is divergent in (4.15), while all derivative terms $f'(0)$ etc. and in particular $f_R(\underline{k})$ are finite and well-defined as $N\to\infty$.

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This leads to a clear candidate for the appropriate scaling of the bare constants a, c, μ^2 and λ: the log-divergence in (4.13) should be absorbed by c, and μ^2 should be determined such that $\tilde{J}_k$ is finite. This allows to have finite a coupling constant λ, which does not require renormalization. Keeping b also finite determines a through the constraint (4.8), rendering the one-point function (4.13) finite and well-defined. Thus we set

$$c = -f(0) + c' \approx -\frac{1}{(8\pi^2\theta)^2}\ln(N) + c' \tag{4.17}$$

where c' is a free finite parameter. Furthermore using

$$\tilde{J}_k = J_k + (i\lambda)^2 c = 8\pi^2\theta(\underline{k}+1) + (4\pi^2\theta^2\mu^2 + (i\lambda)^2 c) \tag{4.18}$$

we choose according to the above discussion

$$\begin{aligned} \mu^2 &= -\frac{(i\lambda)^2}{4\pi^2\theta^2}\, c + \mu_R^2 = \frac{(i\lambda)^2 f(0) - (i\lambda)^2 c'}{4\pi^2\theta^2} + \mu_R^2 \\ &\approx \frac{(i\lambda)^2}{256\pi^6\theta^4}\ln(N) \end{aligned} \tag{4.19}$$

where $\mu_R^2 > 0$ is finite, and a free parameter of the model. This leads to

$$\tilde{J}_k = 8\pi^2\theta(\underline{k} + 1 + \frac{\mu_R^2\theta}{2}), \tag{4.20}$$

which is finite and independent of N.

The parameter b is determined by the renormalization conditions $\langle\phi_{00}\rangle = \langle\tilde{\phi}_{00}\rangle - \frac{\tilde{J}_0}{i\lambda} = 0$ (2.19). At genus 0, this amounts to

$$\sqrt{\tilde{J}_0^2 + 2b} - \tilde{J}_0 + (i\lambda)^2 c' = 0 \tag{4.21}$$

using the above definitions. This has indeed a solution with finite b as long as $|\frac{(i\lambda)^2 c'}{\tilde{J}_0}| < 1$, determining b as a function of c'. Note that $b = O(\lambda)^2$ is always analytic in λ, starting at second order. One particular solution is

$$c' = 0 \quad \Rightarrow \; b = 0 \tag{4.22}$$

where the formulas take a very simple form.

Since $\tilde{J}$ and b are now finite and independent of N, we obtain the renormalized one-point function

$$\langle\phi_{kk}\rangle_{g=0} = \frac{1}{i\lambda}(\sqrt{\tilde{J}_k^2 + 2b} - \tilde{J}_k) + (i\lambda)c' + (i\lambda) f_R(\underline{k}), \tag{4.23}$$

which is finite and well-defined as $N \to \infty$. Note that there is one additional free parameter compared to the 2-dimensional case, given by c'. For the simplest case $c' = b = 0$, this simplifies as

$$\langle\phi_{kk}\rangle_{g=0} = (i\lambda) f_R(\underline{k}) = (i\lambda)\sum_j \frac{1}{\tilde{J}_j}\Big(\frac{1}{\tilde{J}_k + \tilde{J}_j} - \frac{1}{\tilde{J}_0 + \tilde{J}_j}\Big) \tag{4.24}$$

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which coincides with the one-loop result (5.5); we will comment on this fact later. Finally, a is determined by the constraint (4.8),

$$
\begin{aligned}
(\frac{b}{(i\lambda)^2} - a + (i\lambda)^2 c^2/2) &= -\sum_i \frac{1}{\sqrt{\tilde{J}_i^2 + 2b}} =: -g(b) \qquad (4.25)\\
&\approx -\frac{1}{8\pi^2\theta}\int_0^N dx_1 dx_2\, \frac{1}{(x_1+x_2+1)} \approx -\frac{1}{8\pi^2\theta} N \ln N.
\end{aligned}
$$

Therefore

$$
a = g(b) + \frac{b}{(i\lambda)^2} + (i\lambda)^2 c^2/2 = g(0) + (i\lambda)^2 c^2/2 + (finite). \qquad (4.26)
$$

Thus we can trade a for the implicit parameter b, and interpret b as parametrization of a which is determined by the renormalization conditions $\langle\phi_{00}\rangle = 0$.

To summarize, the one-point function is renormalized by requiring the bare parameters c, μ and a to scale as in (4.17), (4.19), and (4.26). This leaves 3 independent finite parameters λ, c' and μ_R of the model. A fourth free parameter b could be introduced by relaxing the condition $\langle\phi_{00}\rangle = 0$.

Diagonal n-point functions By taking higher derivatives of F_0 w.r.t. $\tilde{J}^2$, we obtain the genus 0 contribution to the *connected part of the* n-point functions for diagonal entries $\langle\tilde{\phi}_{i_1 i_1}\dots\tilde{\phi}_{i_n i_n}\rangle$. Note that the (infinite) shift $\tilde{\phi} = \phi + \frac{J}{i\lambda} = \phi + \frac{\tilde{J}}{i\lambda} - (i\lambda)c$ drops out from the connected n-point function for $n \geq 2$, therefore for $n \geq 2$ these coincide with $\langle\phi_{i_1 i_1}\dots\phi_{i_n i_n}\rangle$ and should thus be finite.

Start with the genus 0 contribution. To compute higher derivatives of F_0 w.r.t. $\tilde{J}^2$, we also must take into account the implicit dependence of b on $\tilde{J}^2$. We recall that (4.11) can be written as

$$
\frac{\partial}{\partial b} F_0(\tilde{J}_i^2; b) = -\frac{1}{2}\Big(\frac{b}{\lambda^2} + a + \lambda^2 c^2/2 - \lambda^{-2/3}\, I_0\Big)^2 \qquad (4.27)
$$

which vanishes through the constraint (4.8). Considering this as a function of the independent variables $\tilde{J}_k^2$ and b, we will also encounter

$$
\begin{aligned}
\frac{\partial^2}{\partial b^2} F_0(\tilde{J}_i^2; b) &= -\Big(\frac{b}{\lambda^2} + a + \lambda^2 c^2/2 - \lambda^{-2/3}\, I_0\Big)(1 - \lambda^{-2/3}\frac{\partial}{\partial b} I_0(\tilde{J}^2, b)),\\
\frac{\partial^2}{\partial \tilde{J}_k^2 \partial b} F_0(\tilde{J}_i^2; b) &= \Big(\frac{b}{\lambda^2} + a + \lambda^2 c^2/2 - \lambda^{-2/3}\, I_0\Big)(\lambda^{-2/3}\frac{\partial}{\partial \tilde{J}_k^2} I_0(\tilde{J}^2, b))
\end{aligned}
$$

which still vanishes due to the constraint, while e.g.

$$
\begin{aligned}
\frac{\partial^3}{\partial \tilde{J}_l^2 \partial \tilde{J}_k^2 \partial b} F_0 &= -\Big(\lambda^{-2/3}\frac{\partial}{\partial \tilde{J}_l^2} I_0\Big)(\lambda^{-2/3}\frac{\partial}{\partial \tilde{J}_k^2} I_0)\\
&\quad -\Big(\frac{b}{\lambda^2} + a + \lambda^2 c^2/2 - \lambda^{-2/3}\, I_0\Big)(\lambda^{-2/3}\frac{\partial^2}{\partial \tilde{J}_l^2 \partial \tilde{J}_k^2} I_0)
\end{aligned}
$$

is non-vanishing, and similarly for higher derivatives. We will show below that all these terms are finite as $N \to \infty$ provided λ is small enough.

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Consider the quantities involved in more detail. First, all the

$$I_p(\tilde{J}_k^2, b) = -(2p-1)!!\lambda^{2(2p+1)/3} \sum_i \frac{1}{(\tilde{J}_i^2 + 2b)^{p+\frac{1}{2}}} \tag{4.28}$$

are finite (i.e. convergent as $N \to \infty$) for $p \geq 1$. In particular, we note that $|I_1| < 1$ provided the coupling λ is small enough. Furthermore, the constraint (4.8) implies

$$\frac{\partial}{\partial \tilde{J}_k^2} b = -\frac{\lambda^2}{2(\tilde{J}_k^2 + 2b)^{3/2}} - \lambda^2 \Big(\sum_i \frac{1}{(\tilde{J}_i^2 + 2b)^{3/2}} \Big) \frac{\partial}{\partial \tilde{J}_k^2} b, \tag{4.29}$$

hence

$$\frac{\partial}{\partial \tilde{J}_k^2} b = -\frac{\lambda^2}{2(\tilde{J}_k^2 + 2b)^{3/2}} \frac{1}{1 - I_1} \tag{4.30}$$

which is finite for small $|\lambda|$ since we can assume $|I_1| < 1$ as just shown. Finally, we need

$$\begin{aligned} \frac{\partial}{\partial b} I_p(\tilde{J}, b) &= -\lambda^{-4/3}\, I_{p+1}, \\ \frac{\partial}{\partial \tilde{J}_k^2} I_p(\tilde{J}, b) &= \frac{(2p+1)!!\lambda^{2(2p+1)/3}}{2(\tilde{J}_k^2 + 2b)^{p+\frac{1}{2}}}. \end{aligned} \tag{4.31}$$

which are all finite as $N \to \infty$ provided λ is small enough, for any $p \geq 0$. The same is obviously true for all higher derivatives. Furthermore, we see explicitly that any higher derivative of the r.h.s. in (4.12) w.r.t. $\tilde{J}_k^2$ or b is also manifestly finite. Combining all this, we have shown that all relevant derivatives of F_0 are finite and have a well-defined limit $N \to \infty$, provided λ is small enough, the constraint (4.8) is imposed, and the parameters a, c, μ^2 are renormalized as in (4.17), (4.19), and (4.26). This establishes that the genus 0 contribution to the correlation functions for diagonal entries $\langle \phi_{kk} \dots \phi_{ll} \rangle$ is finite and well-defined.

Higher genus contributions. It is easy to see that the higher genus contributions are also finite under the same conditions. Indeed, in view of the structure of the higher genus contributions F_g stated below (3.12) as found by [2], this follows immediately form the above considerations. Thus we have established

> **Theorem 1** *All derivatives of F_g w.r.t. $\tilde{J}_k^2$ for $g \geq 0$ as well as all F_g for $g \geq 1$ are finite and have a well-defined limit $N \to \infty$, provided λ is small enough, the constraint (4.8) is imposed, and the parameters a, c, μ^2 are renormalized as in (4.17), (4.19), and (4.26).*

The precise condition for λ is that $|I_1| < 1$. The limiting case $I_1 = 1$ will be identified below as an instability of the model.

Since the connected n-point functions are given by the derivatives of $F = \sum_{g \geq 0} F_g$ w.r.t. $\tilde{J}_k^2$, this implies that all contributions in a genus expansion of the correlation functions for diagonal entries $\langle \phi_{kk} \dots \phi_{ll} \rangle$ are finite and well-defined. The general non-diagonal correlation functions are discussed in section 4.3, and also turn out to be finite for arbitrary genus g under the same conditions. Putting these results together we have established renormalizability of the model to all orders in a genus expansion, i.e.

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Theorem 2 *The (connected) genus g contribution to any given n-point function is finite and has a well-defined limit $N \to \infty$ for all g, under the above conditions.*

Moreover, they can in principle be computed explicitly using the above formulas. In particular, since any contribution to F_g has order at least λ^{4g-2}, this implies renormalizability of the perturbative expansion to any order in λ. This is certainly expected in view of the results in [5 − 7].

Note in particular that we did not have to specify whether λ is real, or $i\lambda$ is real, etc. Rather, all genus g contributions are analytic in λ provided $|\lambda|$ is small enough such that $|I_1| < 1$ holds. Therefore we have obtained a definition of the NC ϕ^3 model also for real coupling λ, under the above conditions.

It is also worth pointing out that only the genus 0 contribution requires renormalization, while all higher genus contributions are finite. This is very interesting because the genus 0 contribution can be obtained by various techniques in more general models, see also [11, 12]. This is due to the presence of the oscillator-like potential in the action, which suppress the UV/IR mixing originating from higher genus diagrams.

The parameters μ_R^2, λ and c' are the free moduli of the model, which can be interpreted as mass, coupling, and a further parameter which was introduced by renormalization.

4.2 The 2-point function at genus 0

4.2.1 $\langle\phi_{kl}\phi_{lk}\rangle$.

We can use (2.21) to obtain the genus 0 contribution to the 2-point function $\langle\phi_{kl}\phi_{lk}\rangle$ for $k \neq l$. Using (4.23) we obtain

$$\langle\phi_{kl}\phi_{lk}\rangle_{g=0} = 2\,\frac{\sqrt{\tilde{J}_k^2 + 2b} - \sqrt{\tilde{J}_l^2 + 2b} + (i\lambda)^2\,(f_R(\underline{k}) - f_R(\underline{l}))}{\tilde{J}_k^2 - \tilde{J}_l^2}. \tag{4.32}$$

Note that λ enters also implicitly through b (4.21). For the simplest case $b = 0$, this simplifies to

$$\begin{aligned}\langle\phi_{kl}\phi_{lk}\rangle_{g=0} &= \frac{2}{\tilde{J}_k + \tilde{J}_l} + 2(i\lambda)^2\,\frac{f_R(\underline{k}) - f_R(\underline{l})}{\tilde{J}_k^2 - \tilde{J}_l^2} \\ &= \frac{2}{\tilde{J}_k + \tilde{J}_l} - (i\lambda)^2\,\frac{2}{\tilde{J}_k + \tilde{J}_l}\sum_j \frac{1}{\tilde{J}_j(\tilde{J}_k + \tilde{J}_j)(\tilde{J}_l + \tilde{J}_j)}.\end{aligned} \tag{4.33}$$

The first term is the free propagator, while the second term has somewhat stronger decay properties. This can be rewritten in a more suggestive way using the identity

$$\frac{\langle\phi_{kk} - \phi_{ll}\rangle}{\tilde{J}_k^2 - \tilde{J}_l^2} + \frac{\langle\phi_{kk} + \phi_{ll}\rangle}{(\tilde{J}_k + \tilde{J}_l)^2} = 2\,\frac{\tilde{J}_k\langle\phi_{kk}\rangle - \tilde{J}_l\langle\phi_{ll}\rangle}{(\tilde{J}_k + \tilde{J}_l)^2(\tilde{J}_k - \tilde{J}_l)}, \tag{4.34}$$

which for $b = 0$ and using (4.24), (4.17) leads to

$$\langle\phi_{kl}\phi_{lk}\rangle_{g=0} = \frac{2}{\tilde{J}_k + \tilde{J}_l} - 2(i\lambda)\,\frac{\langle\phi_{kk} + \phi_{ll}\rangle}{(\tilde{J}_k + \tilde{J}_l)^2} + \frac{4(i\lambda)^2}{(\tilde{J}_k + \tilde{J}_l)^2}\Big(\sum_j \frac{1}{(\tilde{J}_k + \tilde{J}_j)(\tilde{J}_l + \tilde{J}_j)} + c\Big). \tag{4.35}$$

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Comparing with the perturbative computation (5.11), this means that the one-loop contribution to the propagator gives the exact result for the genus 0 sector for the case $b = 0$, noting that $\lambda^2 c = \delta J$ using (4.19) and (5.4). This remarkable property can be traced back to (4.9), which is also exact at order λ^2 for $b = 0$. For higher genus, this is no longer the case.

4.2.2 $\langle \phi_{ll}\phi_{kk}\rangle$.

As a further example, consider the 2-point function $\langle \phi_{ll}\phi_{kk}\rangle$ for $k \neq l$, which vanishes in the free case. To compute it from the effective action, we need in principle

$$\langle \tilde\phi_{ll}\tilde\phi_{kk}\rangle - \langle \tilde\phi_{kk}\rangle\langle \tilde\phi_{ll}\rangle = 2i\lambda\frac{\partial}{\partial \tilde J_l^2}2i\lambda\frac{\partial}{\partial \tilde J_k^2}(F_0 + F_1 + \cdots). \tag{4.36}$$

Even though this corresponds to a nonplanar diagram with external legs, it is obtained by taking derivatives of a closed genus 0 ring diagram. Therefore we expect that only F_0 will contribute, and indeed the derivatives of F_1 contribute only to order λ^4. We need

$$\begin{aligned} 2i\lambda\frac{\partial}{\partial \tilde J_l^2}2i\lambda\frac{\partial}{\partial \tilde J_k^2}F_0 &= -\frac{2(i\lambda)^2}{\sqrt{\tilde J_k^2 + 2b}}\tilde J_l^2\frac{\partial}{\partial \tilde J_l^2}\left(\sum_j \frac{1}{\sqrt{\tilde J_k^2 + 2b} + \sqrt{\tilde J_j^2 + 2b}}\right) \\ &= (i\lambda)^2\frac{1}{\sqrt{\tilde J_k^2 + 2b}}\frac{1}{\sqrt{\tilde J_l^2 + 2b}}\left(\frac{1}{\sqrt{\tilde J_k^2 + 2b} + \sqrt{\tilde J_l^2 + 2b}}\right)^2 \end{aligned} \tag{4.37}$$

using again (4.11). Therefore to lowest order we obtain

$$\langle \phi_{ll}\phi_{kk}\rangle = \langle \phi_{kk}\rangle\langle \phi_{ll}\rangle + \frac{(i\lambda)^2}{\tilde J_k\,\tilde J_l}\left(\frac{1}{\tilde J_k + \tilde J_l}\right)^2, \tag{4.38}$$

in complete agreement with the perturbative computation (5.10).

4.3 General n-point functions

Finally we show that all contributions in the genus expansion (and therefore perturbative expansion) of the expectation values of any n-point functions of the form

$$\langle \phi_{i_1 j_1}\ldots.\phi_{i_n j_n}\rangle \tag{4.39}$$

have a well-defined and finite limit as $N \to \infty$ provided b is finite, which means that the model is fully renormalized. The argument is a generalization of the one given in [1], taking into account the degeneracy of J.

In view of (2.16), the insertion of a factor $\tilde\phi_{ij}$ can be obtained by acting with the derivative operator $2i\lambda\frac{\partial}{\partial \tilde J_{ij}^2}$ on $Z(\tilde J^2)$, resp. $F_g(\tilde J^2)$ for fixed genus g. Now any given correlation function of type (4.39) involves only a finite set of indices $i, j, \ldots$. Thus taking derivatives w.r.t. $\tilde J_{ij}^2$ amounts to considering matrices $\tilde J$ of the form

$$\tilde J = \left(\begin{array}{c|c} \mathrm{diag}(\tilde J_1, \ldots \tilde J_k) + \delta\tilde J_{k\times k} & 0 \\ \hline 0 & \mathrm{diag}(\tilde J_{k+1}, \ldots \tilde J_N) \end{array}\right), \tag{4.40}$$

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where k is chosen large enough such that all required variations are accommodated by the general hermitian $k \times k$ matrix

$$\tilde{J}_{k\times k} := \Big(\mathrm{diag}(\tilde{J}_1, \dots \tilde{J}_k) + \delta \tilde{J}_{k\times k}\Big) \tag{4.41}$$

in (4.40), while the higher eigenvalues $\tilde{J}_{k+1}, \dots \tilde{J}_N$ are fixed and given by (4.20). Therefore we can restrict ourselves to this $k \times k$ matrix, which is independent of N. As was shown in section 4.1, all F_g are in the limit $N \to \infty$ smooth (in fact analytic) symmetric functions of the first k eigenvalues squared, hence of the eigenvalues of $(\tilde{J}_{k\times k})^2$. Such a function can always be written as a smooth (analytic) function of some basis of symmetric polynomials in the $\tilde{J}_a^2$, in particular

$$F_g(\tilde{J}_1^2, \dots, \tilde{J}_k^2) = f_g(Tr(\tilde{J}_{k\times k}^2), \dots, Tr(\tilde{J}_{k\times k}^{2k})). \tag{4.42}$$

This can be seen by approximating the analytic function $F_g(z_1, \dots, z_k)$ at the point $z_i = \tilde{J}_i^2$ by a totally symmetric polynomial in the z_i, which correctly reproduces the partial derivatives up to some order n. According to a well-known theorem, that polynomial can be rewritten as polynomial in the elementary symmetric polynomials, or equivalently as a polynomial in the variables $s_n := \sum z_i^n$, $n = 1, 2, \dots, k$. This amounts to the rhs of (4.42).

In the form (4.42), it is obvious that all partial derivatives $\frac{\partial}{\partial \tilde{J}_{ij}^2}$ of F_g exist to any given order, and could be worked out in principle. This completes the proof that each genus g contribution to the general (connected) correlators $\langle \phi_{i_1 j_1} \dots . \phi_{i_n j_n}\rangle$ is finite and convergent as $N \to \infty$. This implies in particular (but is stronger than) renormalizability of the perturbative expansion to any order in λ.

4.4 Approximation formulas for finite coupling

In this section we derive some closed formulas which are appropriate for finite coupling λ, in the large N limit. They will be needed in particular to derive the critical line in section 4.5. For simplicity we only consider the case $b = c' = 0$ (4.22).

We approximate the various sums by integrals. This gives

$$\begin{aligned} I_0 &= -\lambda^{2/3} \sum_i \frac{1}{(\tilde{J}_i^2)^{\frac{1}{2}}} \approx -\frac{\lambda^{2/3}}{8\pi^2\theta} \int_{x_0}^{x_N} dx \int_{y_0}^{y_N} dy \, \frac{1}{(x+y+1)} \\ &\approx -\frac{\lambda^{2/3}}{8\pi^2\theta} N \ln N, \end{aligned} \tag{4.43}$$

where

$$x_n = (n + \frac{1+\mu_R^2\theta}{2}), \quad dx = dn\,. \tag{4.44}$$

Similarly,

$$\begin{aligned} I_1 &= -\lambda^2 \sum_i \frac{1}{(\tilde{J}_i^2)^{\frac{3}{2}}} \approx -\frac{\lambda^2}{(8\pi^2\theta)^3} \int_{x_0}^{x_N} dx \int_{y_0}^{y_N} dy \, \frac{1}{(x+y+1)^3} \\ &\approx -\frac{\lambda^2}{2(8\pi^2\theta)^3} \int_{x_0}^{x_N} dx \frac{1}{(x + (\frac{1+\mu_R^2\theta}{2}) + 1)^2} \approx -\frac{\lambda^2}{2(8\pi^2\theta)^3} \frac{1}{\mu_R^2\theta + 2}. \end{aligned} \tag{4.45}$$

Similarly, all I_p can be approximated by elementary, convergent integrals.

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4.5 Critical line and instability.

We have seen that for small enough coupling $|\lambda|$, the free energy $F = F_0 + F_1 + \cdots$ is regular and finite for any given genus in the renormalized model, since all I_k with $k \geq 1$ are finite *provided* $I_1 \neq 1$.

However, as is manifest in the explicit formulas for F_g at higher genus (3.11) ff., there is a singularity at $I_1 = 1$. Using (4.45), this critical point is given for $c' = b = 0$ by

$$\mu_R^2\theta + 2 = \frac{(i\lambda)^2}{2(8\pi^2\theta)^3}. \tag{4.46}$$

This[3] is similar to the 2-dimensional case [1], and indicates that for the ϕ^3 model with real coupling constant $\lambda' = i\lambda$ stronger than this critical coupling, the model becomes unstable. This is very reasonable, since the potential is unbounded, and the potential barrier around the local minimum becomes weaker for stronger coupling. Therefore this critical line could be interpreted as the point where the quantum fluctuations of ϕ become large enough to see the global instability, so that the field "spills over" the potential barrier. Similar transitions for a cubic potential are known e.g. for the ordinary matrix models, but may also be relevant in the context of string field theory and tachyon condensation [13, 14]. In particular, it is interesting to note that this singularity occurs simultaneously for each given genus, which suggest that some double-scaling limit near this critical point can be taken, again in analogy with the usual matrix models (for a review, see e.g. [15]). Again, such a scaling limit for the Kontsevich model is discussed in [2]. We leave this issue for future work.

5. Perturbative computations

We write the action (2.11) as

$$\begin{aligned}\tilde{S} &= Tr\Big(\frac{1}{4}(J\phi^2 + \phi^2 J) + \frac{i\lambda}{3!}\,\phi^3 - (i\lambda)A\phi\Big)\\ &= Tr(\frac{1}{2}\phi^i_j\,(G_R)^{j;l}_{i;k}\,\phi^k_l + \frac{i\lambda}{3!}\,\phi^3 - (i\lambda)A\phi + \frac{1}{4}(\delta J\phi^2 + \phi^2\delta J))\end{aligned} \tag{5.1}$$

where the finite (renormalized) kinetic term $(G_R)^{j;l}_{i;k} = \frac{1}{2}\delta^i_l\delta^k_j(J^R_i + J^R_j)$ defines the propagator

$$\Delta^{i;k}_{j;l} = \langle\phi^i_j\phi^k_l\rangle = \delta^i_l\delta^k_j\frac{2}{J^R_i + J^R_j} = \delta^i_l\delta^k_j\frac{1/(4\pi^2\theta)}{\underline{i} + \underline{j} + (\mu_R^2\theta + 2)}, \tag{5.2}$$

and we use again the notation (4.14) $\underline{n} = n_1 + n_2$. This corresponds to the finite (renormalized) matrix

$$J^R|n_1, n_2\rangle = 8\pi^2\theta(\underline{n} + 1 + \frac{\mu_R^2\theta}{2})|n_1, n_2\rangle. \tag{5.3}$$

The remaining

$$\delta J|n_1, n_2\rangle = (8\pi^2\theta\frac{\delta\mu^2\theta}{2})|n_1, n_2\rangle \tag{5.4}$$

is part of the counter-term, where $\delta\mu^2 = (\mu^2 - \mu_R^2)$.

[3] Recall that this is obtained imposing the renormalization conditions $\langle\phi_{00}\rangle = 0$ at genus 0. Note also that the rhs is indeed dimensionless.

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1-point function The one-loop contribution to the 1-point function gives

$$\begin{aligned}\langle\phi_{ii}\rangle &= \frac{i\lambda}{J_i^R}A_i - \frac{i\lambda}{2}\frac{1}{J_i^R}\sum_k \frac{2}{J_i^R+J_k^R} \quad + O(\lambda^2)\\ &= -\frac{i\lambda}{J_i^R}\left(-A_i + \frac{1}{8\pi^2\theta}\sum_k \frac{1}{\underline{i}+\underline{k}+2+\mu_R^2\theta}\right) \quad + O(\lambda^2).\end{aligned} \tag{5.5}$$

To proceed, we expand

$$h(\underline{i}) := \frac{1}{8\pi^2\theta}\sum_k \frac{1}{\underline{i}+\underline{k}+2+\mu_R^2\theta} = h(0) + (\underline{i})\,h'(0) + h_R(\underline{i}) \tag{5.6}$$

where

$$\begin{aligned}h(0) &= \frac{1}{8\pi^2\theta}\sum_k \frac{1}{\underline{k}+\mu_R^2\theta+2} \sim \frac{1}{8\pi^2\theta}\,N\log N,\\ h'(0) &= -\frac{1}{8\pi^2\theta}\sum_k \frac{1}{(\underline{k}+\mu_R^2\theta+2)^2} \sim -\frac{1}{8\pi^2\theta}\,\log N,\end{aligned} \tag{5.7}$$

and $h_R(\underline{i})$ is a finite nontrivial function of $\underline{i}$.

We note in particular that the i- dependent term $(\underline{i})\,h'(0)$ in (5.5) forces us to introduce a corresponding counterterm to the action, which we chose to be $A = a + cJ$ (2.10). As discussed in section 2, this is equivalent to an infinite shift (2.9) of ϕ. Taking this into account we have

$$\langle\phi_{ii}\rangle = -\frac{i\lambda}{J_i^R}\left(-(a+cJ_i) + (h(0) + (\underline{i})\,h'(0) + \tilde{h}(\underline{i}))\right) \quad + O(\lambda^2), \tag{5.8}$$

and the condition $\langle\phi_{00}\rangle = 0$ implies

$$\begin{aligned}a + 8\pi^2\theta c\,\frac{\mu^2\theta}{2} &= h(0),\\ c &= \frac{1}{8\pi^2\theta}\,h'(0)\,.\end{aligned} \tag{5.9}$$

This is in complete agreement with (4.17), and also with (4.26) taking into account (5.13). These renormalization conditions guarantee that the one-point function $\langle\phi_{ii}\rangle$ has a well-defined and nontrivial limit $N \to \infty$.

2-point function Next we compute the leading contribution to the 2-point function $\langle\phi_{ll}\phi_{kk}\rangle$ for $l \neq k$, which vanishes at tree level. The leading contribution comes from the nonplanar graph in figure 1, which gives

$$\langle\phi_{ll}\phi_{kk}\rangle = \langle\phi_{kk}\rangle\langle\phi_{ll}\rangle + \frac{1}{4}\frac{(i\lambda)^2}{J_k\,J_l}\left(\frac{2}{J_k+J_l}\right)^2 \tag{5.10}$$

(for $l \neq k$) indicating the symmetry factors, where the disconnected contributions are given by (5.5). This is in complete agreement with the result (4.38) obtained from the Kontsevich model approach. Note that the counterterm δJ does not enter here.

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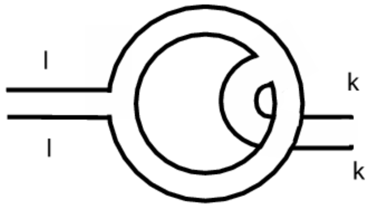

Figure 1: one-loop contribution to $\langle \phi_{ll}\phi_{kk}\rangle$

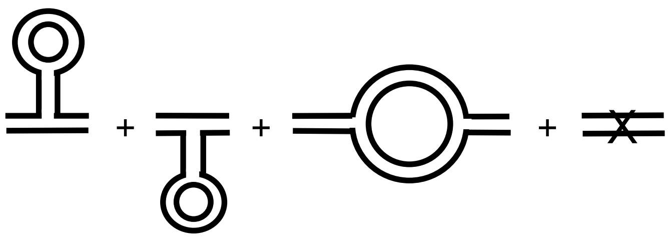

Figure 2: one-loop contribution to $\langle \phi_{kl}\phi_{lk}\rangle$

Similarly, the leading contribution to the 2-point function $\langle \phi_{kl}\phi_{lk}\rangle$ for $l \neq k$, has the contribution indicated in figure 2, but now the counterterm δJ does enter also. This gives the result

$$\begin{aligned}\langle \phi_{kl}\phi_{lk}\rangle &= \frac{2}{J_k^R + J_l^R} - 2(i\lambda)\frac{\langle \phi_{kk} + \phi_{ll}\rangle}{(J_k^R + J_l^R)^2} \\ &+ \frac{4}{(J_k^R + J_l^R)^2}\Big(\sum_j \frac{(i\lambda)^2}{J_k^R + J_j^R}\frac{1}{J_l^R + J_j^R} - \frac{\delta J_l + \delta J_k}{2}\Big) + O(\lambda^4) \qquad (5.11)\end{aligned}$$

The first term is the free propagator, the next term the tadpole contributions, and the last them the one-loop contribution in figure 2 with counterterm δJ.

We have to adjust the parameters such that the result is well-defined and nontrivial. The last term is logarithmically divergent,

$$\begin{aligned}\sum_j \frac{1}{J_k^R + J_j^R}\frac{1}{J_l^R + J_j^R} &\approx \frac{1}{(8\pi^2\theta)^2}\sum_j \frac{1}{(\underline{j} + \mu_R^2\theta + 2)^2} \\ &= -\frac{1}{8\pi^2\theta}\,h'(0) \sim \frac{1}{(8\pi^2\theta)^2}\,\log N. \qquad (5.12)\end{aligned}$$

Therefore the divergent terms $-\frac{\delta J_l + \delta J_k}{2} - (i\lambda)^2\frac{1}{8\pi^2\theta}\,h'(0)$ must cancel, i.e.

$$\begin{aligned}-8\pi^2\theta\frac{\delta\mu^2\theta}{2} &= (i\lambda)^2\frac{1}{8\pi^2\theta}\,h'(0), \\ \delta\mu^2 &\sim \frac{(i\lambda)^2}{256\,\pi^6\theta^4}\,\log N \qquad (5.13)\end{aligned}$$

in complete agreement with (4.19). Hence the mass is log-divergent as expected, and no wavefunction-renormalization Z is required.

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We note in particular that these one-loop computations already give the exact results for the renormalization, as found using the Kontsevich model. This reflects the "super-renormalizability" of the model, which is thus established rigorously.

6. Discussion and conclusion

We have shown that the selfdual NC ϕ^3 model in 4 dimensions can be renormalized and essentially solved in terms of a genus expansion, by using the Kontsevich model. This provides closed expressions for the free energy and certain n-point functions for each genus, which are finite after renormalization and valid for finite nonzero coupling. Remarkably, the genus 0 contribution turns out to be exact at one loop in a special point $b = c' = 0$ of moduli space. An instability is found if the coupling constant reaches a critical coupling, as expected for the ϕ^3 model.

It is very remarkable that a nontrivial 4-dimensional NC ϕ^3 field theory allows such a detailed analytical description. There is no commutative analog to our knowledge, which shows that the noncommutative world in some cases is more accessible to analytical methods than the commutative case. Furthermore, these NC ϕ^3 field theories in different (even) dimensions can all be mapped to the same Kontsevich model $Z(M)$, for different M with different eigenvalues and degeneracies. While the techniques used in this paper are more-or-less restricted to the ϕ^3 interaction, it is worth pointing out that the renormalization is determined by the genus 0 contribution only, which is accessible in a wider class of models; see also [11, 12] in this context.

Perhaps the main gap in our treatment is the lack of control over the *sum* over all genera g. While the contributions for each genus are manifestly analytic in the coupling constant λ, we have not shown that the sum over g converges in a suitable sense. This would amount to a full construction of the model. However, it seems very plausible that this is the case, and the sum defines an analytic function in λ near the origin. It should be possible to establish this using the relation with the KdV hierarchy or the relation with topological gravity, which is beyond the scope of this paper. Another interesting extension would be the case $\Omega \neq 1$, which could be considered as a perturbation around $\Omega = 1$, rather than around $\Omega = 0$. We hope to study this problem in the near future.

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